Let's take another tricky probability question today and employ two different methods to solve it.

Question: Two couples and one single person occupy a row of five chairs at random. What is the probability that neither couple sits together (the husband and the wife should not occupy adjacent seats)?

(A) 1/5
(B) 1/3
(C) 3/8
(D) 2/5
(E) 1/2

Solution: You can approach this GMAT problem in different ways. One way is a step-by-step case evaluation. Another is to go the reverse way: count all the arrangements in which at least one couple sits together and subtract that from the total arrangements possible. My method of choice is generally the second one. The only catch is that you have to remember to subtract from the total number of arrangements.

What is the total number of arrangements (without any restrictions)? I hope you remember your basic counting principle and will agree that it should be 5! (Five people arranged in five seats). Now, let's find out the number of favorable cases.

We will discuss both the methods in detail.

Method 1: Step – by – Step Case Evaluation

Let's say the two couples are {C1h, C1w} and {C2h, C2w} and the single person is S.

Case 1: S takes the first/last chair

If S takes the first chair, any one of the remaining people can take the chair next to him (4 ways). Say, C1h takes this spot. The next place cannot be occupied by C1w but either of C2h and C2w can occupy it (2 ways). Say, C2h occupies it. The fourth chair can be occupied by only one of the remaining 2 people since C2w cannot take it now (1 way). The last chair has only one person remaining for it.

Total number of acceptable arrangements in which S takes the first chair = 4*2*1 = 8

The case would be exactly the same if S took the fifth seat. Think of it this way: The chairs have seat numbers 1-5. Now the numbers have reversed, 1 switched with 5, 2 switched with 4 and 3 is as it is. Now S is sitting on seat number 5 and we have exactly 8 more arrangements possible.

Total number of acceptable arrangements in which S takes the first or fifth chair = 8*2 = 16

Case 2: S takes the second/fourth chair

If S takes the second seat, any of the remaining four people could sit next to S on either side. However, we need to ensure that both people sitting on either side of S are not a couple i.e. C1h, S, C1w should not occupy the first, second and third seats respectively because then C2h and C2w are left and only 2 adjacent seats are vacant. But they cannot take adjacent seats. This means that there are 4 ways in which the first seat can be occupied — i.e., anyone can take it but there are only 2 ways in which the third seat can be occupied since the person taking the third seat must be from the other couple — i.e., if C1h takes the first seat, only C2h or C2w could take the third one. Now we have 2 people and 2 seats leftover. Fourth seat can be occupied in only one way since if C2h takes the third seat, C2w cannot take it i.e. one of the remaining two people cannot take it. Thereafter, one person and one seat are leftover so the fifth seat can be occupied in one way.

Total number of acceptable arrangements in which S takes the second chair = 4*2*1*1 = 8

The case would be exactly the same if S took the fourth seat.

Total number of acceptable arrangements in which S takes the second or fourth chair = 8*2 = 16

Case 3: S takes the middle seat i.e. the third seat

If S takes the third seat, there are two seats on his left and two on his right. We have to ensure that a couple doesn't sit on one side and the other side would automatically be couple-free. Any one of the four people can occupy the first seat (say C1h takes it). The second seat can be taken by one person from the other couple i.e. by C2h or C2w so it can be occupied in only 2 ways. Now we have two people leftover and two seats. Either one of them could take the fourth seat so it can be occupied in 2 ways. The fifth seat can be occupied in one way.

Total number of acceptable arrangements in which S takes the third chair = 4*2*2*1 = 16

Total number of favorable arrangements = 16 + 16 + 16 = 48

Total number of arrangements = 120

Probability that neither couple sits together = 48/120 = 2/5

Answer (D)

Method 2:

The logic I use here is the one we use to solve SETS questions. It needs a little bit of thought but minimum case evaluations.

There are two couples. We don't want either couple to sit together. Let's go the reverse way – let's make at least one of them sit together. We can then subtract this number from the total arrangements to get the number of arrangements in which neither couple sits together.

Would you agree that it is easy to find the number of arrangements in which both couples sit together? It is. We will work on it in a minute. Let's think ahead now.

How about 'finding the number of ways in which one couple sits together?' Sure we can easily find it but it will include those cases in which both couples are sitting together too. But we would have already found the number of ways in which both couples sit together. We just subtract 'both couples together' number from 'one couple together' number and get the number of arrangements in which ONLY one couple sits together. Think of SETS here.

Let's do this now.

Number of arrangements in which both couples sit together: Let's say the two couples are {C1h, C1w} and {C2h, C2w} and the single person is S. There are three groups/individuals. They can be arranged in 3! ways. But in each couple, husband and wife can be arranged in 2 ways (husband and wife can switch places)

Hence, number of arrangements such that both couples are together = 3!*2*2 = 24

Number of arrangements such that C1h and C1w are together: C1 acts as one group. We can arrange 4 people/groups in 4! ways. C1h and C1w can be arranged in 2 ways (husband and wife can switch places).

Number of arrangements in which C1h and C1w are together = 4! * 2 = 48
But this 48 includes the number of arrangements in which C2h and C2w are also sitting together.
Therefore, number of arrangements such that ONLY C1h and C1w sit together = 48 – 24 = 24
Similarly, number of arrangements such that ONLY C2h and C2w sit together = 24

Number of arrangements in which at least one couple sits together = 24 + 24 + 24 = 72

Number of arrangements in which neither couple sits together = 120 – 72 = 48

Probability that neither couple sits together = 48/120 = 2/5

I believe that the second method is much faster and easier. Nevertheless, it’s good to know and understand both.